CSUC
Department of Physics
301A Mechanics
***PORTFOLIO* Problem 3b.)** due:

I. APPROXIMATING THE AN-HARMONIC PENDULUM

Introduction: Somewhere early-on in your physics-life you learned that a "Simple Pendulum" executes "Simple-Harmonic-Motion" and you understood very well that, as a consequence of merely *being* one of those marvelous "S.H.O. appearances in natures", the period of oscillation should be *Amplitude Independent.* Ah, but *you* know it couldn't be quite true! After all, everybody in *Introductory Mechanics* kept warning you to use *small amplitudes (!)* ... just to keep it "accurate". Well ... then ... so what *is* the truth ? Again, the full answer turns out to be a pretty tough problem. Yes, but again, we don't need the *full* answer ... a highly accurate approximation will do very nicely. This PORTFOLIO problem will lead you through two procedures to get the answers you need.

Let's get started by just noting some simple observations.

a) First, this **is** a conservative problem which means we can just write down the conservation of energy equation: (*kinetic energy*) + (*potential energy*) = *constant* ... and go from there.

b) The real difficulty was representing the potential energy. Yes, it's $m \cdot g \cdot h$, but we need to use the angle of swing, which we'll call θ, as our descriptive variable. If the pendulum has length ℓ and we have a point-mass m on the end, then the full *true* potential energy (taking zero at the bottom of the swing) is given by $m \cdot g \cdot \ell \; (1 - cos(\theta)\,)$.

c) If the angle involved is *small* (i.e. $\ll 1$), it makes great sense to express the cosine term as a power series in θ .

$$cos(\theta) \;=\; 1 \;-\; \frac{\theta^2}{2!} \;+\; \frac{\theta^4}{4!} \;... \tag{1}$$

Remember, the angle is measured in radians so that θ really is a very small number compared to 1 for angles less than, say, ten degrees (check this!). Then it makes sense to keep the first non-vanishing variable term and toss the rest. But suppose we now make our starting angle *"somewhat bigger"* ... what correction do we introduce then? Show yourself that the correction introduced by the next "quartic" term really is still smaller, but in an infinite series with alternating terms we are guaranteed that adding the next tern *does* improve the approximation. This will be our next correction. Here's the trick. We are going to add in a term written as $\lambda \cdot \theta^4$. In our case we will, (at the end of the problem, to be sure) actually set the *value* of λ as $1\,/\,4!$. But, throughout the analysis, we will treat λ as a variable and that allows us to do a (Taylor) expansion in the variable λ. This is a sweet trick. Use it early and often.

d) Once we are convinced that an added effect is *"small"*, we can include this small intrusion on our basic problem as a ***perturbation*** to be included in some consistent order. The added math is generally far simpler (and much more enlightening) than the formal solution itself. You will do this two different ways ! That they agree precisely (after what seems like a world of algebra ...) is a delicious confirmation that the level of *truthful certainty* inherent in modern mathematical rigor does, indeed, tie everything together "behind the scenes". We put up with the pedantic (and not a little daunting ...) nature of abstract mathematical proof precisely because of the great power of this certainty.

1. *Method # 1. Energy. Of course, ... we start by using Natural Units.*

We raise the mass m up to its starting angle θ_{max} and then release it from rest as a pendulum. This problem has only four natural sizes: $\{g,\, \ell,\, m,\, \theta_{max}\}$. As in so many gravitational problems the mass drops out leaving only the length ℓ, the starting angle and the acceleration of gravity as natural and "characteristic" sizes in the problem.

Question Set 1.) Use ℓ as your unit of length L i.e. $\ell \,=\, 1 \cdot L$ and g as your unit of acceleration i.e. $g \,=\, 1 \cdot L\,/\,T^2$. First find the unit of time T and the unit of velocity L/T. Next, write down the ***Active*** conservation of energy equation in these units. Notice how constants seemingly *disappear* in ***Active*** equations. It will have the form:

$$\frac{1}{2}\left(\dot{\theta}\right)^2 \;=\; \mathcal{U}_{max} \;-\; \mathcal{U} \tag{2}$$

You will, of course, substitute in the specific "quartic" model we are now using for the potential energy. Now separate the variables θ and t as we always do, to arrive at:

$$\frac{d\,\theta}{\sqrt{\mathcal{U}_{max} \;-\; \mathcal{U}}} \;=\; \pm\sqrt{2}\;dt \tag{3}$$

Inside the radical is the variable λ ! You are to perform a Taylor expansion in λ up to **first order** which leaves you with a simplified equation which you must now integrate as the *"Second Newtonian Integral"*. You will actually see a pair of integrals to do (i.e. the "zeroth" and "first order" terms). **Don't freak out here!** These integrals are actually very do-able (oh **yes** they are ...) ! The first use of these integrated expressions is simple. If the angle proceeds in time from its initial value of θ_{max} down to zero (i.e. the bottom), then the time expended is $T_{1\,/4}$, i.e. one-quarter of the full period of motion. Perform the integration over this full interval and you have the first relation we seek! It tells you (**correct to first order in λ**) how the period has changed by including the (true) quartic potential energy. Write down this result and savor it! It is what you have been "missing" your whole (young) physics-life. Evaluate to what percentage the period changes if the starting angle is even as large as $\theta_{max} \;=\; \pi/2$. Could you notice this in the laboratory ? The final problem staring at you now is that your result from doing the integrals to an **arbitrary intermediate angle** will give you the full answer to the "motion" relationship in the form of "time" as a function of "angle". You will now need to *invert* this to give us "angle" as a function of "time". This is to say, the integrals explicitly give us our true relationship in the form $t \;=\; \tilde{t}\,(\theta)$ ... and we wish to have it in the form $\theta \;=\; \tilde{\theta}\,(t)$. Remember, you need only perform this inversion *correct to first order in* λ ! The best way to do this is to use "implicit differentiation" on the relationship in front of you (recall, we did this earlier in the semester). You will be able to extract the quantity: $\partial\,\theta(t,\;\lambda)/\partial\lambda|^{\lambda\,=\,0}$ which will allow you to write explicitely:

$$\theta(t,\;\lambda) \;\approx\; \theta_o(t) \;+\; \lambda^1\cdot\partial\,\theta(t,\;\lambda)/\partial\lambda\;|^{\lambda\,=\,0} \;+\; \text{higher power terms} \tag{4}$$

So now you know how the angle depends on time for every angle. Once again, the bottom of the swing is one quarter of the full periodic cycle and the angle is zero there. So now you know it must be the case that:

$$0 \;=\; \theta_o(T_{1/4}) \;+\; \lambda^1\cdot\partial\,\theta(T_{1/4},\;\lambda)/\partial\lambda\;|^{\lambda\,=\,0} \tag{5}$$

From this you may, yup ... once again, find $T_{1/4}$ and once again you need only do it *correct to first order in* λ ! The answer will agree (it **must** agree! ... this is mathematics ...) with the first result you just derived. I think you can see that we have used this theme of "power series expansion" repeatedly in this problem. It is very (very !) powerful and will be one of your first "go to" tools throughout your physics life and beyond.

2. ***Method # 2. Newton. Of course, ... we start by using Natural Units ... again!***

This problem also allows a solution proceeding directly from Newton's second law. The solution process really "feels" different and the equations themselves really do "look" different. But mathematics is obdurate. It cannot be bent. It links everything " behind the scenes" ... and the answers come out the same. By the way, this might help convince you that, with mathematics at our side, we are never really "out of ammunition". The number of approaches is beyond counting ... so we have merely to be *more imaginative!* Truth will out ... and we can find it!

Question Set 2.) Write down the conservation of energy equation in its ***Active*** form from above (including the quartic term, of course). Now take the total time derivative of this whole equation and, once you have canceled off the common overall factor of $\dot{\theta}$ which appears, you will have the *Newton Second Law* form of the equations of motion for this problem in our chosen coordinates. It will look like:

$$\ddot{\theta} \;=\; -\,\theta \;+\; \lambda \cdot \theta^3 \tag{6}$$

Our expansion parameter λ has a slightly different "value" this time ... but plays exactly the same role as before. We wish to express its effect on the problem in a Taylor's series. To do this we write the full solution $\theta(t,\ \lambda)$ as a power series as follows:

$$\theta(t,\ \lambda) \;\approx\; \theta_o(t) \;+\; \lambda^1 \cdot \theta_1(t) \;+\; \text{higher power terms} \tag{7}$$

Now we insert this in Newton's Second Law (equation (6) just above) on **both** sides of the equal sign. The theorem on the **uniqueness** of power series now allows us to equate individual terms of equal powers in λ but on opposite sides of the equal sign, one by one. This gives us, then, a whole sequence of **individual** equations ... one for each power. We observe that each succeeding equation depends on the solution of the preceding equation. These equations "fold" on each other. This is something new and subtle. Wow.

We achieve first, then, for the "zeroth power" term:

$$\ddot{\theta}_o(t) \;=\; -\,\theta_o(t) \tag{8}$$

This is the zeroth order equation and describes a *Simple Harmonic Oscillator* as expected. But the new *"first order"* equation that follows is:

$$\ddot{\theta}_1(t) \;=\; -\,\theta_1(t) \;+\; \lambda \cdot \theta_o(t)^3 \tag{9}$$

This is also a S.H.O. situation but with a novel *"Inhomogeneity"* on the right hand side involving $\theta_o(t)$. This is where the magic is. These equations also have initial conditions. Since we assume $\theta(t,\ \lambda)$ starts from rest at angle θ_{max}, We can satisfy these conditions to all orders by specifying that: $\theta_o(t\,=\,0) \,=\, \theta_{max}$ and $\dot{\theta}_o(t\,=\,0) \,=\, 0$ for the zeroth order function, and then $\theta_1(t\,=\,0) \,=\, 0$ and $\dot{\theta}_1(t\,=\,0) \,=\, 0$ for the first order function.

Solving equation (8) is something you know how to do. All the new content we wish to examine is found in equation (9) ! What you need to recognize is that this is something you have seen before! It appeared before your eyes when we studied the "causal response" of Simple Harmonic Oscillators to a driving force that gets *"turned on"* at some particular moment. One of the key mantras of physics is that **"the same equations have the same solutions"**. You are to write down the solution to equation (8), and then stick that into our previously discovered solution to equation (9) (... and for which you just might have to look back in your notes and find!) .

This result brings us back to the setting of **Question Set 1.)** . From the solutions to equations {8 & 9} you are to write down the *"first order corrected"* equation for the motion and the period. They **must !** , of course, agree with our earlier results.